On Spatial Formulation of the Orr-Sommerfeld Equation for Thin Liquid Films

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Krantz and Owens (1973) obtained an approximate solution to the Orr-Sommerfeld equation for the problem of linear stability of a thin liquid film. In their analysis, the disturbances are taken to be traveling waves which grow or decay exponentially in space. Comparisons of their theoretical results with some of the experimental results of Krantz and Goren (1971) and the theoretical results obtained by Anshus and Goren (1966) for temporally growing disturbances led them to conclude:

> The solution to the spatial formulation of the Orr-Sommerfeld equation compares more favorably with wave property data than do solutions to classical temporal formulation of this stability problem.

The above quotation is taken from the abstract of the work by Krantz and Owens.

In this comment, we show that the above conclusion was reached through an inappropriate use of Gaster's (1963, 1965) theorem. We also take this opportunity to clarify some important concepts in film instability.

Gaster (1963) proves rigorously that for a traveling disturbance the temporal amplitude growth rate β_i is related to the spatial growth rate $-\alpha_i$ by the relation

$$\beta_i = -\frac{\partial(\alpha_r C_r)}{\partial \alpha_r}\alpha_i \qquad (1)$$

In the above equation, C_r is the phase velocity of the disturbance and $\alpha_r = 2\pi h_0/\lambda$ defines the wave number where h_0 and λ are respectively the film thickness and the wave length. $\partial(\alpha_r C_r)/\partial\alpha_r = C_g$ is the group velocity (Yih, 1959) with which the energy of the wave propagates. For nondispersive waves C_r is independent of α_r, and for a monochromatic dispersive wave C_r depends only on the given wave length and thus is a constant. Therefore, the group velocity and the phase velocity are identical for either a monochromatic wave or nondispersive waves. Note that the temporal neutral curve $\beta_i = 0$ and the spatial neutral curve $\alpha_i = 0$ are identical since $C_g \neq 0$ on the neutral curve. Equation (1) is obtained, however, under the following two conditions:

1. $|\alpha_i(\beta_i)_{\max}| << C_g$,
2. For the wave of a given length, wave speeds and the amplitude growth rates are independent of the wave amplitude.

The first condition is known to hold in boundary layer flows according to Shen's (1954) computation and will be shown presently to hold also in liquid film flows investigated by Krantz and Owens. The second condition can be satisfied only during the initial stage of instability when the nonlinear dependence of wave properties on wave amplitude remains small. It is this second condition which is not completely satisfied in Krantz and Owens' computation as shown below.

According to the notation of Krantz and Owens $\beta_i = \alpha_r C_i$ where C_i is the imaginary part of the complex wave speed. The value of $(\beta_i)_{\max}$ for the film flow considered by Krantz and Owens can be obtained from the curve denoted by A-G in their Figure 2 and is approximately equal to 0.09. According to the data given in their Figure 3, $|\alpha_i| < 0.05$. On the other hand, the reported wave speed never falls below 2.3 and thus we have

$$|\alpha_i(\beta_i)_{\max}| << C_r$$

Hence, the condition (1) is satisfied. Note that the measured wave properties correspond to disturbances of controlled frequencies, that is, monochromatic waves and thus $C_g = C_r$. The curve denoted by O-K in their Figure 5 is obtained from the solution to the Orr-Sommerfeld equation for spatially growing disturbances. The curve denoted by A-G in the same figure is obtained from the temporal solution given in Figure 2 by use of the transformation given by Equation (1). The two curves coincide only near the neutral curve where $\alpha_i < 0.1$. For other values of α_i, two curves differ considerably. Moreover, the curve obtained from the spatial formulation compares more favorably with the experimental points as shown in Figure 5. This led Krantz and Owens to reach the above quoted conclusion. However, for the experimental points given in Figure 5 the corresponding wave amplitudes are already comparable to the film thickness, and thus the condition 2 is violated. Therefore, for these values of α_i the application of the Gaster transformation is not justifiable and the comparison of the curve A-G with the experimental points is meaningless. The poor comparison between the curve A-G and the experimental points merely shows that the Gaster transformation is inapplicable to nonlinear waves but does not prove the inferiority of the temporal formulation. The coincidence of curves A-G and O-K near the neutral curves is quite to be expected since both conditions 1 and 2 are fully satisfied near the neutral curve, and the Gaster theorem guarantees this coincidence. Why, then, does the amplification curve obtained from the spatial formulation of the linear stability problem compare so well with experiments? Does not this prove the superiority of spatial formulation? We merely point out that in the solution of Krantz and Owens it is assumed that the parabolic velocity profile in the primary flow can be replaced by a constant. It remains to be seen if the good comparison will remain when this assumption is removed. The method of solving the Orr-Sommerfeld equation without this assumption for finite values of wave number and Reynolds number is available (Graef, 1966). We further point out that the wave speeds predicted by use of the spatial formulation as well as the temporal formulation are consistently lower than the observed values. Citing Lin's (1969) work among others, Krantz and Owens attribute this difference to the neglected finite amplitude effect which always raises the wave speed predicted by linear theory. Unfortunately, this is inaccurate. As is demonstrated by Lin (1971), the wave speed can actually decrease as amplitude increases due to a capillary effect for relatively short waves. This would further widen the discrepancy between theories and experiments on wave speeds.

We conclude this comment by pointing out that the temporal and spatial formulations of linear stability of parallel flows with respect to traveling dispersive or non-disturbances are equivalent within the framework of linear theory if the conditions 1 and 2 are met. The application of the linear theory to the nonlinear wave regime only leads to confusion and does not predict wave properties (wave speed, amplitude, length and growth rate) quantitatively. Quantitative understanding of film stability must be based on nonlinear stability theories. Unfortunately, the existing nonlinear theories are rather limited in scope. For example, the theory of Lin (1969, 1970, 1974) is valid only for $\alpha_r << 1$ and comparisons of his theory with the observations of Krantz and Goren (1971) for finite values of α_r are not even possible. Finally, we mention the work of Agrawal (1972) on the nonlinear stability of liquid films with respect to spatially growing disturbances.

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NOTATION

C_g = group velocity
C_i = imaginary part of the complex wave velocity
C_r = real part of the complex wave velocity
h_0 = mean film thickness

Greek Letters

α_r = wave number = $2\pi h_0/\lambda$
α_i = negative of spatial amplification rate
β_i = temporal amplification rate = $\alpha_r C_i$
λ = wave length

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Additional Comments on the Spatial Formulation of the Orr-Sommerfeld Equation for Thin Liquid Films

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In the previous note Lin (1974) has presented further discussion on a recent paper of Krantz and Owens (1973). The latter developed an approximate closed-form analytical solution to the spatial formulation of the Orr-Sommerfeld equation appropriate to falling film flow. Their results, when compared with the wave property data of Krantz and Goren (1971), suggest that the spatial formulation is superior to the temporal formulation of the Orr-Sommerfeld equation for this flow. Lin is critical of these results. His principal comments concerning the work of Krantz and Owens are summarized here:

1. Lin asserts that Krantz and Owens' conclusions were based on an inappropriate use of Gaster's (1965) theorem in that the data used by Krantz and Owens involved wave properties and in particular growth rates which were dependent on the wave amplitude; that is, Lin claims that these data did not apply to the initial stages of growth. Furthermore Lin states that "... for the experimental points in Figure 5 [of Krantz and Owens (1973)] the corresponding wave amplitudes are already comparable to the film thickness, and thus the condition 2 [of Lin (1974)] is violated."

2. Lin suggests that the quantitative agreement between the data and the solution of Krantz and Owens might be an artifact of this approximate method of solution.

3. Lin maintains that the finite amplitude effect could decrease rather than increase the wave speed as suggested by Krantz and Owens. In this note we will briefly reply to each of these comments of Lin.

Lin begins by reviewing the arguments of Gaster (1965) leading to the relation between the spatial and temporal amplification rates. These arguments were omitted from the paper of Krantz and Owens but were included in the thesis of Owens (1972). This development of Lin is instructive and perhaps should have been included in the paper of Krantz and Owens. We wish to add, however, that in order to arrive at Equation (1) in Lin's note, it is necessary to assume that the disturbances